B.Sc Part I (Hons)
Paper III Paper IV

Examples of Leibnitz theorem.

Example 1. If $y = e^{a\sin^{-1}x}$ Prove that

(i) $(1-x^2)y_2 - xy_1 - a^2y = 0$

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$

Solution: (i) Here $y = e^{a\sin^{-1}x}$

$$\therefore y_1 = e^{a\sin^{-1}x} \times \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2}\,y_1 = ay \quad \therefore (1-x^2)y_1^2 = a^2y^2$$

Again differentiating

$$(1-x^2)2y_1y_2 + y_1^2(-2x) = a^2 2yy_1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = a^2y$$

$$\therefore (1-x^2)y_2 - xy_1 - a^2y = 0 \quad \text{--- (1)}$$

Proved

(ii) Again according to Leibnitz theorem differentiating (1) n times, we get

$$\{(1-x^2)y_{n+2} + {}^nc_1(-2x)y_{n+1} + {}^nc_2(-2)y_n\} - \{xy_{n+1} + {}^nc_1 y_n\} - a^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{1\cdot 2}(-2)y_n - xy_{n+1} - ny_n - a^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} + y_{n+1}(-2nx - x) + y_n\{-n(n-1) - (n+a^2)\} = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$$

Proved

Example (2) If $y = \sin(m\sin^{-1}x)$, prove that

(i) $(1-x^2)y_2 - xy_1 + m^2y = 0$.

(ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

(iii) $\lim_{x \to 0} \frac{y_{n+2}}{y_n} = n^2 - m^2$.

Solution:— ~~We have~~ Here $y = \sin(m\sin^{-1}x)$

$\therefore y_1 = \cos(m\sin^{-1}x) \times m \cdot \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow \sqrt{1-x^2}\, y_1 = m\cos(m\sin^{-1}x)$

$\Rightarrow (1-x^2) y_1^2 = m^2\cos^2(m\sin^{-1}x)$

$= m^2[1-\sin^2(m\sin^{-1}x)] = m^2(1-y^2)$

Now differentiating.

$(1-x^2)\, 2y_1 y_2 + y_1^2(-2x) = m^2(-2yy_1)$

$\Rightarrow (1-x^2) y_2 - xy_1 = -m^2 y$

$\therefore (1-x^2) y_2 - xy_1 + m^2 y = 0$

For (ii) part.

$(1-x^2) y_2 - xy_1 + m^2 y = 0$

Differentiating this according to Leibnitz theorem, we get

$(1-x^2) y_{n+2} + {}^nc_1(-2x) y_{n+1} + {}^nc_2(-2) y_n$

$+ \{x y_{n+1} + {}^nc_1(1) y_n\} + m^2 y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - 2nx y_{n+1} - n(n-1) y_n - x y_{n+1} - x y_n$

$+ m^2 y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} + \{m^2 - n(n-1) - n\}$

$y_n = 0$

$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2 - n^2) y_n = 0$

Proved.

Now taking the limit of (1) as $x \to 0$, we get

$y_{n+2} = (n^2 - m^2) y_n$

i.e. $\lim_{x \to 0} \frac{y_{n+2}}{y_n} = n^2 - m^2$

Proved!

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